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LETTER TO THE EDITOR

Eigenvalue statistics of disordered conductors

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Abstract. We apply the statistical measures of Wigner, Dyson and Mehta to quantum-mechanical systems having intrinsic disorder. We observe a transition from regular Poisson-like to Wigner-like eigenvalue statistics, and relate these two limiting behaviours to the ballistic and mesoscopic regimes of quantum transport. In strongly disordered systems we observe that the eigenvalue spectra have a more complex structure, whose nature seems to provide a useful indicator of transport behaviour. We also observe similar effects in the spectra of quantum-mechanical systems whose classical analogues exhibit chaotic behaviour, and we can therefore provide a semi-quantitative description of the non-universal conductance fluctuations recently observed by Marcus and co-workers in ballistic microstructures.

Since the realization by Wigner [1] that the spectra of complex nuclei could be interpreted in terms of the eigenvalue properties of random matrices, the study of such matrices has become an active area of research [2, 3]. In particular, it has been shown that the statistical properties of the eigenvalues show fluctuations that are anomalously small, so that, for example, the variance of the number of eigenvalues in a given range of energy depends only logarithmically on the value of the range [4]. Such statistical properties have frequently been employed to identify systems that share the same universal properties as the random matrices of Dyson and Mehta.

One such area of current interest is the study of quantum-mechanical disordered systems, where recent work [5] has emphasized statistical properties analogous to those of systems which are classically chaotic [6]. Appropriate descriptions can therefore be given either via semi-classical methods or directly in terms of the theory of random matrices. However, much of this work has been motivated by the universal behaviour [7] of transport properties, not on the statistics of the eigenvalues of the disordered Hamiltonians. A particular exception is work on the 'kicked rotator', which can be mapped onto the disordered Anderson Hamiltonian in one dimension, and where the localization index of the wavefunctions can be related to the character of the eigenvalue spectrum [8].

The universal behaviour of transport properties refers to conduction in disordered systems in the 'mesoscopic' regime, where the mean free path, ℓ, is smaller than the typical size of the system, L, but where the coherence length ξ of the electron wavefunctions still exceeds L. This is the regime of weak localization, where as shown by Al'tshuler [7] and later by Lee and co-workers [9], the conductance g of the samples exhibits universal fluctuations of amplitude Δg close to e^2/h. The connection of these fluctuations to the statistics of eigenvalues was first made by Al'tshuler and Shklovskii [10], but in subsequent

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work their direct approach has been superseded by use of the statistical properties of transfer matrices [11]. In terms of numerical work, however, there are significant advantages to the statistical analysis of eigenvalues, since larger and more complex systems can be employed when eigenfunctions and/or matrix elements are not required.

In this letter, we study disordered mesoscopic systems using numerical investigations of their eigenvalue statistics. In the usual regime, $\ell < L < \xi$, we observe entirely Wigner-like spectra, corresponding to the existence of conductance fluctuations of universal amplitude. In the limit when the disorder is very strong, we observe behaviour which is energy dependent, displaying transitions from regular (Poisson-like) to Wigner-like behaviour, and which suggests a description for conductance fluctuations of non-universal amplitude. As an application of this description, we also analyse the eigenvalue statistics of 'stadia': systems in the ballistic regime. We show for the first time that the boundary scattering in these systems has an effect on the eigenvalue statistics essentially indistinguishable from the effects of disorder, and suggest that non-universal conductance fluctuations should therefore be observed. Recent experiments on nanostructure devices designed to resemble 'stadia' [12] lend support to our point of view.

The inspection of the eigenvalue spectrum is performed using the Δ_3 statistic of Dyson and Mehta [4], which is related to the variance $\Sigma^2(E, E_s)$ of the number of eigenvalues in the range E_s [13]. Following a conventional procedure [14], we renormalize the eigenvalue spectrum so that it has a mean spacing of one unit, and then define $N(E)$ to be the number of renormalized eigenvalues below energy E. $\Delta_3(E, E_s)$ is the variance of the best linear fit to $N(E)$ over a range of E_s units commencing at E, averaged over a range E_R of E :

$$\Delta_3(E, E_s) = \left\langle \mathrm{Min}_{A,B} E_s^{-1} \int_E^{E+E_s} \left[N(E', E_s) - A - BE'\right]^2 \mathrm{d}E' \right\rangle_{E_R}.$$

There are well known results for Δ_3 in the case when there is no dependence on E, namely when the eigenvalue sequence is homogeneously distributed. When the E_i are taken from a random distribution in the range $[0, n]$, so that the spacing of adjacent eigenvalues has a Poisson distribution as in a regular (integrable) system [15]

$$\Delta_3^{\text{Poisson}}(E, E_s) = \tfrac{1}{15} E_s.$$

When the E_i are the eigenvalues of a random matrix having time-reversal invariance belonging to the Gaussian orthogonal ensemble (GOE)

$$\begin{aligned} \Delta_3^{\text{GOE}} &\simeq \pi^{-2}(\log E_s + 0.0687) \quad && E_s \geq 0.5 \\ &\simeq \tfrac{1}{15} E_s && E_s \ll 1. \end{aligned}$$

This logarithmic GOE behaviour is also the signature of quantum systems whose classical analogues are chaotic [2].

Figure 1 shows our results for rectangular samples, where the simple tight-binding Hamiltonian with diagonal disorder is defined on a square mesh with $\simeq$ 5000 mesh points. In units of the nearest-neighbour matrix element, the diagonal site energies are either $+10$ or 0, chosen randomly with probabilities c and $1 - c$ respectively. In cases where there are two subbands, only the one at lower energy is used for analysis. The curve for $c = 0.0$ is the case of no disorder and is included for completeness: the other curves show $\Delta_3(E, E_s)$ for the range $c = 0.001$ to 0.008. Each data set is averaged over the entire energy range, since for $E_s < 25$ units there is only a weak dependence of $\Delta_3(E, E_s)$ on E, the position in the band [16]. Using a range $E_R = 200$ units we find a variation in Δ_3 of around ± 0.1 for $E_s \simeq 20$.

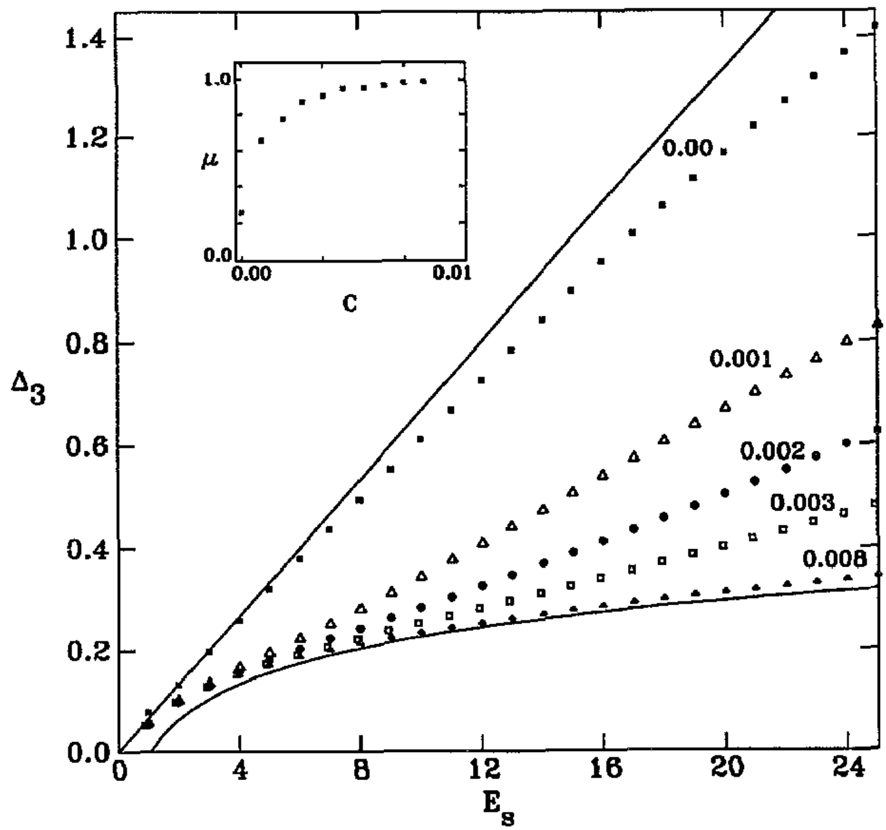

Figure 1. $\Delta_3(E, E_s)$ as a function of E_s for weakly disordered systems. The curves are labelled with values of the concentration c. All the curves are averages over a complete data set, as described in the text. The inset shows the order parameter μ as a function of c.

The data show a clear transition from Poisson to GOE behaviour. Such a transition has been observed elsewhere in a variety of different quantum systems [3, 17], where the corresponding classical systems display regular (integrable) and chaotic behaviour respectively, and where the spectra can be fitted with an order parameter μ, which represents the eigenvalues as superpositions of independent Poisson and GOE sequences. Such an analysis [18] gives

$$\Delta_3(E, E_s) = \Delta_3^{\text{Poisson}}[E, (1-\mu)E_s] + \Delta_3^{GOE}(E, \mu E_s)$$

where μ is the fraction of eigenvalues belonging to the chaotic GOE sequence(s). Although we do not propose a semi-classical analysis of our tight-binding models, we adopt μ as a useful (dis)order parameter for our data: it is shown as a function of c in the inset to figure 1. We see no advantage in adopting the parametrization of Izrailev [8], since it is the least reliable in our regime of interest.

In the language of (quantum) transport theory, the data of figure 1 correspond to the ballistic regime $L < \ell < \xi$, where conductance fluctuations, although present, do not have the universal amplitude. However, since the data for $c \simeq 0.008$ is close to the limit $L \simeq \ell$, where the universal amplitude is first obtained, a further increase in the parameter c produces data which for some ranges of energy are in the mesoscopic regime, $\ell < L < \xi$.

Typical behaviour of Δ_3 in such a case is shown in the upper part of figure 2 for $c = 0.15$: it is qualitatively different, showing a systematic transition from GOE behaviour, $\mu \simeq 1$, in the centre of the band to Poisson behaviour, $\mu \simeq 0$, at the band edges. Comparison with computations of the conductivity in similar models [19, 20] suggests that this is a transition from extended states, which show the characteristic conductance fluctuations of universal amplitude, to localized states which exhibit insulating behaviour. Our preliminary calculations of the participation ratio for the wavefunctions in smaller systems confirm this suggestion, which is also in accord with the analysis of Izrailev [8], whose localization length is essentially equivalent, within a normalization factor, to the participation ratio.

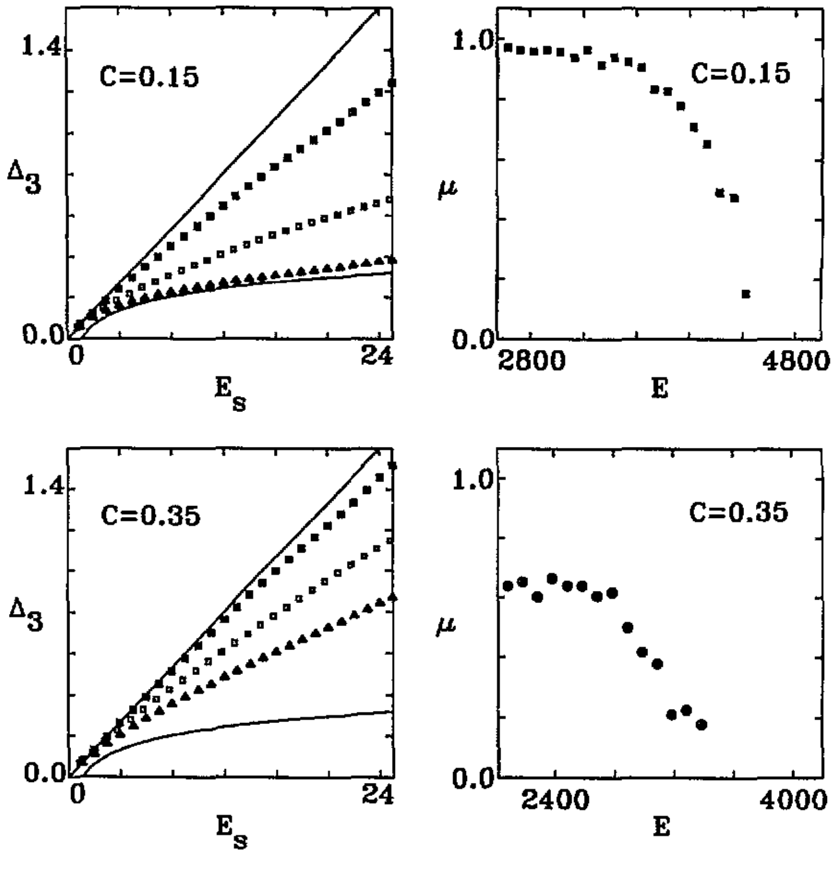

Figure 2. $\Delta_3(E, E_s)$ as a function of E_s for different energy ranges and μ as a function of E for $c = 0.15$ and $c = 0.35$. The data are ensemble averages over 10 samples with different representations of the disorder. Δ_3 is shown for values of E just above the centre of the band, half way between the centre and the top of the band, and near the top of the band, and in each case is averaged over a range $E_R = 200$ units. μ is obtained for ranges $E_R = 100$ units, as described in the text.

An original and informative way to describe this behaviour is to plot μ as a function of E. This is also shown in the upper part of figure 2: each data point represents an average over a range $E_R = 100$ units. Varying E_R does not change the data in any significant way, and thus we conclude that the distribution of the eigenvalues changes from GOE to Poisson in a gradual manner. The analysis shows no sign of an abrupt mobility edge: μ varies gradually with energy in a manner closer to that of Δg than to that of the participation ratio [19].

Further information is presented in the lower part of figure 2, where $c = 0.35$, such that the disorder is 'strong' and $\ell < \xi < L$. The behaviour of Δ_3 shows clearly that this regime corresponds, again, to a superposition of GOE and Poisson distributions, but here the value of μ never reaches its maximum value of unity. It is also known that in this regime Δg no longer has a universal amplitude, even in the centre of the band where g itself shows metallic behaviour [19]. Indeed, Δg varies with energy in much the same way as the parameter μ, so that it seems appropriate to use the value of μ to predict the value of Δg. If only those states obeying GOE statistics contribute to the fluctuations, in a natural extension of Al'tshuler and Shklovskii's original idea [10], then we might replace the density of states at the Fermi energy, $N(E_F)$, which occurs in the Kubo expression for the conductivity, by $\mu N(E_F)$, and write $\Delta g \simeq \mu^2$. This semi-quantitative relationship is also consistent with the data of Harris and Houari [20], computed for a tight-binding Hamiltonian with various types of substitutional and topological disorder in two dimensions, and where Δg is around 0.1 times g itself.

We further suggest that this relationship provides a description for the non-universal conductance fluctuations observed in the ballistic regime, and in particular in the nanostructures of Marcus and co-workers [12]. To illustrate this idea, we have computed the eigenvalue statistics of a set of 'stadia' that interpolate between a rectangle and a conventional half-stadium. The rectangle has vertices at $(\pm L, L)$ and $(\pm L, L - \sqrt{3}L)$, and the half stadium is bounded by the rectangle and also by the inscribed semi-circle of radius L, centred at $(0, 0)$. Intermediate stadia cover the region interior to the rectangle and to semi-circles of different radii R, all centred at $(0, 0)$, but varying in radius from $R = \sqrt{2}L$ (the rectangle) to $R = L$ (the stadium), as shown in the inset to figure 3. They are specified by a parameter $p = \sqrt{2} - R/L$. The analysis is carried out by discretizing the wave equation on a square mesh, and computing its eigenvalues by direct solution of the matrix eigenvalue problem, effectively solving a tight-binding model on a finite lattice. The mesh size is chosen so that there are $\simeq 5000$ odd-parity eigenvalues in the finite band.

The results are shown in figure 3. Only eigenvalues corresponding to states of odd parity with respect to reflections about symmetry axes are used, and as in the case of the disordered systems, each data set is averaged over the entire energy range. To make contact with the data of Marcus *et al*, we propose that their stadia (and for that matter their circles) are not completely in the GOE regime, so that $\mu < 1$. There are a number of reasons why this might be so. One possibility is that their stadia are not perfect, but resemble our intermediate structures. This seems consistent with their observation that the effective areas of their structures are some 30% smaller than shown in the micrographs. If for example we choose $p \simeq 0.0$, then $\mu \simeq 0.3$, which gives a value of Δg around 10% of the universal value, consistent with the experiments. It is also possible that the potential distributions inside the experimental stadia are distorted by long-range Coulomb potentials from charged impurities located outside the stadia. Such a situation would also give rise to reduced amplitude for the conductance fluctuations, much as in the presence of rough edges to the samples [21].

In conclusion, we have shown that in many important respects, the behaviour of electrons is the same both in conventional disordered environments, and in surroundings leading to classical chaos. Differences appear only when the disorder is sufficiently large that samples are in the true mesoscopic regime. We also propose that an analysis of the electronic eigenvalue spectrum in terms of the Wigner–Dyson–Mehta statistics is a valuable tool for the investigation of transport in ballistic and mesoscopic systems, as well as for systems

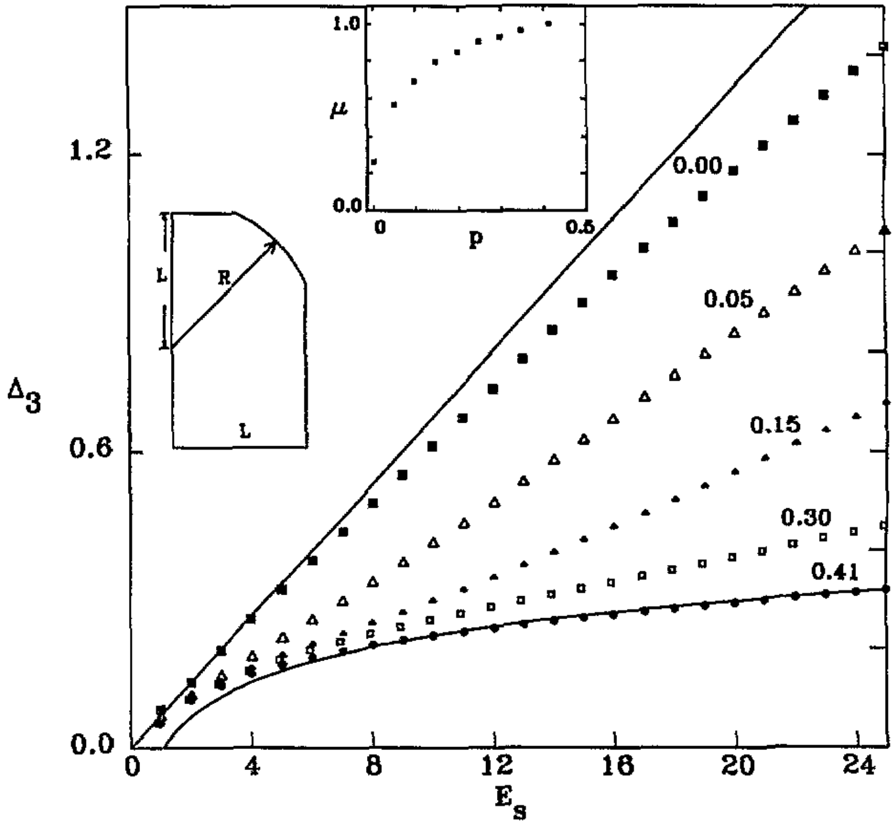

Figure 3. $\Delta_3(E, E_s)$ as a function of E_s for systems intermediate between the stadium and the rectangle. The curves are labelled with values of the parameter $p = \sqrt{2} - R/L$. All the curves are averages over a complete data set, as described in the text. One inset shows the geometry of (one half of) a stadium, the other shows the order parameter μ as a function of p.

beyond the mesoscopic regime, with the order parameter μ having a close association with the amplitude of the conductance fluctuations Δg. Semi-quantitative agreement with the data of Marcus and co-workers suggests that detailed analysis of the relationship between μ, Δg and g itself would be most productive. Such investigations are ongoing.

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